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Mixing at the interface between two fluids in porous media: a boundary-layer solution

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A lighter fluid (fresh water) flows steadily above a body of a standing heavier one (sea water) in a porous medium. If mixing by transverse pore-scale dispersion is neglected, a sharp interface separates the two fluids. Solutions for interface problems have been derived in the past, particularly for the case of interest here: sea-water intrusion in coastal aquifers. The Péclet number characterizing mixing, $Pe = b'/\alpha_T$, where b' is the aquifer thickness and α_T is transverse dispersivity, is generally much larger than unity. Mixing is nevertheless important in a few applications, particularly in the development of a transition layer near the interface and in entrainment of sea water within this layer. The equations of flow and transport in the mixing zone comprise the unknown flux, pressure and concentration fields, which cannot be separated owing to the presence of density in the gravity term. They are nonlinear because of the advective term and the dependence of the dispersion coefficients on flux, the latter making the problem different from that of mixing between streams in laminar viscous flow.

The aim of the study is to solve the mixing-layer problem for sea-water intrusion by using a boundary-layer approximation, which was used in the past for the case of uniform flow of the upper fluid, whereas here the two-dimensional flux field is non-uniform. The boundary-layer solution is obtained in a few steps: (i) analytical potential flow solution of the upper fluid above a sharp interface is adopted; (ii) the equations are reformulated with the potential and streamfunction of this flow serving as independent variables; (iii) boundary-layer approximate equations are formulated in terms of these variables; and (iv) simple analytical solutions are obtained by the von Kármán integral method. The agreement with an existing boundary-layer solution for uniform flow is excellent, and similarly for a solution of a particular case of sea-water intrusion with a variable-density code. The present solution may serve for estimating the thickness of the mixing layer and the rate of sea-water entrainment in applications, as well as a benchmark for more complex problems.

1. Introduction

Flow of a lighter liquid overlying a heavier one in porous media occurs in a few important applications, e.g. sea-water intrusion in coastal aquifers, flow of fresh water above brine bodies at the bottom of deep formations, upconing due to pumping of fresh water underlain by salt water or to pumping of oil floating above water. We are interested primarily in the fresh-water sea-water case and will refer in the following to these two fluids, of densities ρ_f and ρ_s, respectively. Furthermore, the study is limited to steady flow.

These flow patterns are stable and a common approximation is to regard the two fluids as separated by a sharp interface. Then, the problem becomes one of free-surface flow and solutions have been derived in the past analytically, either exactly or by using various approximations (see, for a review, Bear 1979), and numerically (see e.g. Bakker 2003). If only the lighter fluid is in motion and the heavier one is at rest, the problem is mathematically similar to that of a free surface separating a liquid and air.

In reality, a transition zone develops between the two fluids owing to mixing by molecular diffusion and by transverse pore-scale dispersion. The mixing is responsible for the drawing of sea water into coastal aquifers (which is part of the so-called submarine groundwater discharge, Smith 2004) and its flushing by fresh water. Also, mixing is responsible for salinity of water pumped by wells, even if their screens are sufficiently far above the sea-water body.

The intensity of mixing between sea water and fresh water is characterized by the Péclet number, $Pe = VL/D_T$, where V is a fresh-water velocity scale, L is a length scale (e.g. thickness of the aquifer) and D_T is the transverse dispersion coefficient (molecular diffusion is generally negligible). The latter can be written as $D_T = V\alpha_T$, where α_T is the transverse pore-scale dispersivity. In most conceivable applications, $Pe = L/\alpha_T$ is generally much larger than unity.

Modelling of flow in presence of mixing is difficult as it is influenced by density variations, and as a result it is not possible to separate the flow and transport equations. Most of the solutions in the past were obtained by numerical methods and various codes and semi-analytical approximations were developed for this purpose. A development based on a perturbation scheme is presented by Dentz *et al.* (2006), who also provide a comprehensive review of the literature.

In all these studies, relatively small values $Pe = O(10^1\text{–}10^2)$ were assumed. This choice was motivated primarily by numerical constraints, namely the need to render numerical dispersion smaller than the actual one. As a result, relatively thick transition zones and high rates of flushing were present in the solutions and the salient question is whether the adopted values of α_T were realistic.

The experimental determination of α_T under laboratory conditions and homogeneous media was carried out in the past (see e.g. List & Brooks 1967) and it led to very small values $\alpha_T = (10^{-1}\text{–}10^{-2})d$, where d is the pore scale. Determining α_T at the large scale pertaining to natural conditions, is difficult. Analysis of large-scale tracer transport experiments (Fiori & Dagan 1999) arrived at the surprisingly small values $\alpha_T \approx 0.5$ mm. Although somewhat larger values up to $\alpha_T = 32$ mm were reported in the literature (Rugner *et al.* 2004), the associated Pe numbers are still very large. Our interest in the problem was motivated by measurements of salinity profiles in the Yarkon–Taninim aquifer in Israel (Paster, Dagan & Guttman 2006), which revealed the existence of a relatively narrow mixing zone between fresh and salt waters in spite of the large scale and the high heterogeneity of this formation. Hence, it seems that the numerical codes used so far are of limited applicability in solving problems characterized by the high, but realistic values $Pe = O(10^3\text{–}10^4)$.

Similar problems, of a thin transition zone, are encountered in various fields of fluid mechanics and the appropriate tool to tackle them is the boundary-layer (BL) approximation, which simplifies the problem considerably and makes possible the derivation of accurate analytical or numerical solutions. This approximation was used in order to solve a similar problem of the viscous flow of two parallel streams of different velocities (e.g. Lock 1951).

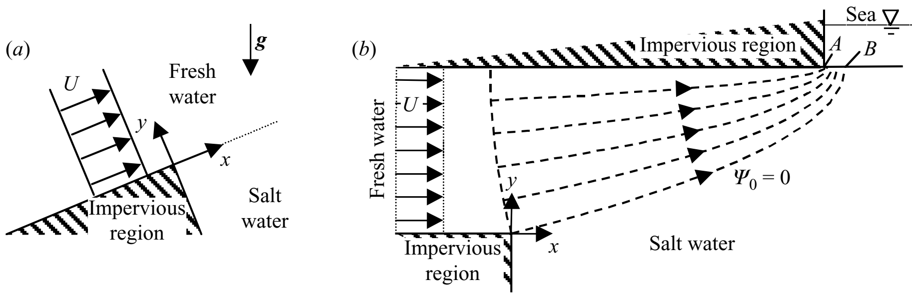

FIGURE 1. (*a*) The geometry of the problem solved by Van Duijn & Peletier (1992). (*b*) The geometry of the sea-water intrusion problem in a confined coastal aquifer of thickness b'. The dashed lines are streamlines of the Glover (1959) solution.

A few attempts have been made to apply the BL approximation to the salt-water intrusion problem (Dagan 1971; Rubin 1983). However, a complete and consistent approach was applied by Van Duijn & Peletier (1992). In their pioneering work, they solved the simple case of uniform flow of fresh water above a planar interface separating it from salt water (figure 1*a*). Though this is an idealized configuration, the results are of interest in establishing the procedure and in gaining insight into the mixing mechanism. However, the findings are not directly applicable to the more realistic conditions of non-uniform flow, such as that depicted in figure 1(*b*). The aim of the present study is to generalize the Van Duijn & Peletier (1992) approach to such flows, focusing on the two-dimensional sea-water intrusion problem, in which fresh water flows in a confined aquifer toward the sea, whereas the sea-water body lies beneath it.

The plan of the paper is as follows. The problem is stated mathematically first, the sharp interface solution is presented next, the problem is then reformulated in the ϕ (potential), ψ (streamfunction) plane, the BL equations are derived and the latter are subsequently solved by using the von Kármán integral approach. Illustration of the results and comparison with previous solutions conclude the paper.

2. Mathematical statement of the problem

The following equations describe the general problem of the steady flow of a mixture of variable density of salt and fresh water (see, for instance, Dagan 1989).

Darcy's law (representing the balance between pressure gradient, viscous stresses and gravity) is

$$\boldsymbol{q}' = -\frac{\kappa}{\mu}(\nabla p' - \rho' \boldsymbol{g}), \tag{2.1}$$

where $\boldsymbol{q}'$ is the fluid volumetric flux (specific discharge), p' is the pressure, ρ' is the density, $\boldsymbol{g}$ is the gravity acceleration vector, κ is the isotropic permeability and μ is the coefficient of viscosity. The equation of mass conservation of the mixture is given by

$$\nabla \cdot (\rho' \boldsymbol{q}') = 0. \tag{2.2}$$

The salt transport is governed by the mass balance equation

$$\nabla \cdot (C' \boldsymbol{q}') = \nabla \cdot \boldsymbol{F}'_c, \tag{2.3}$$

where C' is the concentration (mass of salt/volume of fluid) and $\boldsymbol{F}'$ is the dispersive flux. The latter is related to the concentration gradient by

$$(F'_c)_i = n(D'_0\,\delta_{ij} + D'_{ij})\frac{\partial C'}{\partial x'_j}, \tag{2.4}$$

where n is the porosity, D'_0 is the effective coefficient of molecular diffusion, D'_{ij} is the tensor of pore-scale dispersion and δ_{ij} is the Kronecker unit tensor.

To close the system, a state equation $\rho' = funct(C')$ is required. The equations may be simplified in the case of weak concentrations, as pertinent to fresh and sea waters, for which $(\rho_s - \rho_f)/\rho_f \simeq 0.025$. Hence, we adopt the following widely used assumptions: the viscosity coefficient μ is constant in (2.1), the Boussinesq approximation, i.e. ρ' variations are neglected in (2.2) and they interact with flow through the gravity term of (2.1) only, in the state equation ρ' depends linearly on the concentration C' (Holzbecher 1998, § 2.2) and the dispersion coefficient depends on the flux as in the case of a tracer. With these simplifications, we rewrite (2.2)–(2.4) as follows

$$\nabla \cdot \boldsymbol{q}' = 0, \tag{2.5}$$

$$\boldsymbol{q}' \cdot \nabla \rho' = \nabla \cdot \boldsymbol{F}', \tag{2.6}$$

$$F'_i = n(D'_0\,\delta_{ij} + D'_{ij})\frac{\partial \rho'}{\partial x'_j}, \quad nD'_{ij} = \alpha_T q'\delta_{ij} + (\alpha_L - \alpha_T)\frac{q'_i q'_j}{q'}, \tag{2.7}$$

where α_T and α_L are the transverse and longitudinal dispersivities, respectively, and q' is the flux modulus. Equations (2.1) and (2.5)–(2.7) form a closed system for $\boldsymbol{q}'$, p' and ρ'. We assume that the medium is homogeneous and isotropic, i.e. κ, α_T and α_L are constant. Subsequently, we switch to dimensionless variables with the aid of a fresh water flux U and a length scale L as follows: $x = x'/L$, $y = y'/L$, $\boldsymbol{q} = \boldsymbol{q}'/U$, $\boldsymbol{F} = \boldsymbol{F}'/[U(\rho_s - \rho_f)]$, $p = p'/[(\rho_s - \rho_f)gL]$, $\rho = (\rho' - \rho_f)/(\rho_s - \rho_f)$, $\gamma = [\kappa g(\rho_s - \rho_f)]/(U\mu)$, $\varepsilon = Pe^{-1} = \alpha_T/L$, $\lambda = \alpha_L/\alpha_T$, $D_0 = nD'_0/(UL)$. Furthermore, we eliminate the pressure p' by applying the rot operator to (2.1) to arrive at the following system for $\boldsymbol{q}$ and ρ

$$\nabla \times \boldsymbol{q} = \gamma \boldsymbol{k} \times \nabla \rho, \tag{2.8}$$

$$\nabla \cdot \boldsymbol{q} = 0, \tag{2.9}$$

$$\boldsymbol{q} \cdot \nabla \rho = \nabla \cdot \boldsymbol{F}, \tag{2.10}$$

$$F_i = \left\{ D_0\delta_{ij} + \varepsilon\left[q\delta_{ij} + (\lambda - 1)\frac{q_i q_j}{q} \right] \right\} \frac{\partial \rho}{\partial x_j}, \tag{2.11}$$

$$\text{that is,}\quad \boldsymbol{F} = (D_0 + \varepsilon q)\nabla\rho + \varepsilon(\lambda - 1)\frac{\boldsymbol{q}}{q}(\boldsymbol{q} \cdot \nabla\rho), \tag{2.12}$$

where $\boldsymbol{k}$ is a unit vector in the upward vertical direction. Equations (2.8)–(2.11) were the starting point of Van Duijn & Peletier (1992). Unlike the case of a tracer, the solution of the flux cannot be separated from that of the density, owing to the presence of ρ in (2.8). Furthermore, the system is nonlinear, owing to both advective and dispersive fluxes in (2.10) and (2.11). The parameters D_0 and ε are much smaller than unity, calling for a boundary-layer solution of this system.

3. Sharp interface solution

Since D_0 and ε (2.11) are very small, their neglect leads to the zero-order approximation $\boldsymbol{F}=0$, i.e. $\boldsymbol{q}_0 \cdot \nabla \rho_0 = 0$ in (2.10). The ensuing sharp interface solution is the one pertinent to two fluids, with $\rho = 0$ in the fresh water and $\rho = 1$ in the salt water. For the sake of completeness we review here a few such solutions.

It can be seen from (2.8) and (2.9) that the fresh-water flux satisfies

$$\nabla \times \boldsymbol{q}_0 = 0, \quad \nabla \cdot \boldsymbol{q}_0 = 0, \tag{3.1}$$

and in two-dimensional potential and streamfunction can be defined by

$$\boldsymbol{q}_0 = \nabla \phi_0 = \nabla \psi_0 \times \mathbf{1}_z, \quad \nabla^2 \phi_0 = \nabla^2 \psi_0 = 0. \tag{3.2}$$

In the case of salt water at rest, the two fluids are separated by a streamline and integration of (2.8) across the interface, for ρ discontinuous, leads to the relationships

$$q_{0f} = \gamma \sin\theta_{0f}, \quad \sin\theta_{0f} = q_{0yv}/q_{0f}, \tag{3.3}$$

where q_{0f} is the fresh-water flux tangent to the interface, q_{0yv} is its vertical component, and θ_{0f} is the angle between the interface and the horizontal axis.

We consider in this study two particular cases: uniform flow and sea-water intrusion.

3.1. *Fresh-water uniform flow* (*figure 1a*)

The boundary condition for the fresh-water flow is $q_0 = q_{0x} = 1$, $q_{0y} = 0$ for $x = 0$, $y > 0$. It follows from (3.3) that the solution $q_{0x} \equiv 1$, $q_{0y} \equiv 0$ for $x > 0$, $y > 0$ is possible if the angle between the x-axis and the horizontal direction satisfies

$$\sin\theta_{0f} = 1/\gamma. \tag{3.4}$$

Hence, a discontinuity of ρ and $\boldsymbol{q}$ prevails at the interface $y = 0$, streamlines being parallel to the x-axis.

In view of the developments of the next section, it is worth discussing the meaning of (3.4) in terms of the pressure gradient along the interface. The salt water being at rest, it is seen from Darcy's Law (2.1) that the pressure gradient component parallel to the interface balances the salt-water weight component in the same direction. Since pressure is continuous across the interface, the same pressure gradient balances the lighter fluid weight supplemented by the frictional force that is proportional to q_{0x}. It is easy to ascertain that for a stratified flow for which ρ is a function of y only and it decreases continuously from $\rho = 1$ to $\rho = 0$, a similar exact solution with $q_{0y} \equiv 0$ and (3.4) exists if

$$q_{0x} + \rho = 1 \tag{3.5}$$

across the transition zone.

3.2. *Sea-water intrusion* (*figure 1b*)

This is a schematic representation of aquifer fresh-water flow toward the sea, the heavier sea water intruding into the aquifer from below. The boundary conditions are of no flow $q_{0y} = 0$ at the lower $x' < 0$, $y' = 0$ and upper $x' < x'_A$, $y' = b'$ boundaries of the confined part, respectively, whereas constant pressure prevails at the sea bottom $x' > x'_A$, $y' = b'$. At $x' \to \infty$, $y' < b'$, hydrostatic pressure prevails in the salt-water body. Fresh-water flow is uniform, of flux $U = Q'/b'$ at the inlet $x' \to -\infty$, Q' being the fresh-water discharge. We adopt $L = b'$ and U as length and flux scales and switch to dimensionless variables. Along the interface $y = y(x)$, condition (3.4) prevails at each point i.e.

$$q_{0f} = \gamma \sin\theta_{0f}, \tag{3.6}$$

$$\sin\theta_{0f} = q_{0y}/q_{0f} = \frac{\mathrm{d}y/\mathrm{d}x}{[1 + (\mathrm{d}y/\mathrm{d}x)^2]^{1/2}}, \tag{3.7}$$

x being horizontal and y vertical, upward.

The problem has been solved exactly by using the hodograph method (Bear & Dagan 1964). The impervious bottom was assumed to extend over the entire x-axis, but this is immaterial since the sea water is at rest. Although the derivation of the exact solution requires a numerical quadrature, it was found (Bear & Dagan 1964, figure 3) that for the dimensionless parameter $\eta = 1/\gamma = Q'/[(\kappa g/\mu)(\rho_s - \rho_f)b'] < 0.5$, the solution is accurately approximated by the simple one derived by Glover (1959) for $b' \to \infty$. We adopt the latter, which covers most cases of interest in applications, as it implies a length of penetration greater than the aquifer thickness. Thus, the Glover solution is given by $Z = (1 + \mathrm{i}\eta)\Omega_0 - (1/2)\eta\Omega_0^2$ where $Z = x + \mathrm{i}y$, $\Omega_0 = \phi_0 + \mathrm{i}\psi_0$ and $\mathrm{i} = (-1)^{1/2}$. Separation of real and imaginary parts leads to

$$x = \phi_0 - \eta\psi_0 + \tfrac{1}{2}\eta\left(\psi_0^2 - \phi_0^2\right), \quad y = \psi_0 + \eta\phi_0(1 - \psi_0). \tag{3.8}$$

Streamlines, and in particular the interface defined by $\psi_0 = 0$, are confocal parabolas. The fresh-water flow takes place (figure 1*b*) between the interface $y = 1 - (1 - 2\eta x)^{1/2}$ ($\psi_0 = 0$) and the top $y = 1$ ($\psi_0 = 1$). The seepage face (figure 1*b*) extends over $x_B - x_A = \eta/2$ while the distance to the toe is $x_A = 1/(2\eta) - \eta/2$. The flux magnitude at any point and along the interface is given by

$$q_0 = [(1 - \eta\phi_0)^2 + \eta^2(1 - \psi_0)^2]^{-1/2} \quad (0 < \phi_0 < 1/\eta,\ 0 < \psi_0 < 1), \tag{3.9a}$$

$$q_{0f} = [(1 - \eta\phi_0)^2 + \eta^2]^{-1/2} \quad (0 < \phi_0 < 1/\eta), \tag{3.9b}$$

respectively. It is emphasized that at the origin (the toe), the flux has the finite value $q_{0f} = [1 + \eta^2]^{-1/2}$ (which is close to unity), and the interface is not tangent to the bottom since $\mathrm{d}y/\mathrm{d}x = \eta$ at $x = y = 0$. This local inconsistency stems from adopting the Glover (1959) solution and the discrepancy diminishes as $\eta \to 0$.

In the present study, we regard the flow pattern (3.9) in the domain defined by the equipotentials $\phi_0 = 0$ and $\phi_0 = 1/\eta$ and the streamlines $\psi_0 = 0$ and $\psi_0 = 1$, which are mapped on the (x, y)-plane by (3.8), as representing the actual flow above the interface for the configuration of figure 1(*b*).

4. Reformulation of the mixing problem in the (ϕ_0, ψ_0) plane and the derivation of the boundary-layer equations

No matter how small are D_0 or ε, the right-hand side of (2.11) becomes large at the transition zone owing to the gradient of ρ, similar to the case of viscous shear flow between two streams or past a body. This is a typical BL problem and Van Duijn & Peletier (1992) were the first to tackle it, while maintaining the density effect in Darcy's law. However, their solution, to be discussed below, was derived for the idealized case of uniform flow (figure 1*a*). Our aim is to solve the sea-water intrusion problem (figure 1*b*) and to establish a procedure that can be applied to mixing in other cases of non-uniform flow. This extension is similar to the passage from the viscous BL for flow past a plate to the flow past a body of finite thickness. Along these lines, we adopted the procedure developed by Cole

(1968, chap. 4.2) for this purpose. The BL approximation is incorporated in a singular perturbation expansion, with ϕ_0, ψ_0 (the potential and the streamfunction of the potential flow) rather than x, y, serving as independent variables (a similar procedure has been applied for mixing of a tracer, Dagan 1971). The advantage of this general formulation for viscous flow past bodies is that it is applicable to any two-dimensional flow without separation. Toward its extension for our case, the first step is to rewrite the equations of flow and mixing, (2.8)–(2.11), in terms of ϕ_0, ψ_0 as independent variables. This implies extending the mapping of the (x, y)-plane onto the (ϕ_0, ψ_0)-plane slightly beneath the sharp interface. This may pose some delicate problems in the general case in the toe region, but is simple for the case investigated here (figure 1*b*).

In order to reformulate (2.8)–(2.11) we use the standard expressions of the operator ∇ in an orthogonal curvilinear system (see e.g. Batchelor 1967, Appendix 2). With the metric coefficients $h_1 = h_2 = 1/q_0$ in the present case, the flow equation (2.8) becomes

$$\frac{\partial}{\partial \phi_0}\left(\frac{q_\psi}{q_0}\right) - \frac{\partial}{\partial \psi_0}\left(\frac{q_\phi}{q_0}\right) = -\gamma\left(\frac{\cos\theta_0}{q_0}\frac{\partial \rho}{\partial \phi_0} - \frac{\sin\theta_0}{q_0}\frac{\partial \rho}{\partial \psi_0}\right), \tag{4.1}$$

where q_ϕ and q_ψ are the components of $\boldsymbol{q}$ along the streamlines and equipotentials of the $\boldsymbol{q}_0$ field, respectively. Similarly, $\theta_0 = \sin^{-1}(\boldsymbol{q}_0 \boldsymbol{\cdot} \boldsymbol{k}/q_0)$ is the angle between the tangent to a streamline and the horizontal direction. The solution of the sharp interface problem is supposed to provide q_0 and θ_0 as functions of ϕ_0, ψ_0. In particular, along the sharp interface $\psi_0 = 0$, the relationship (3.7) between q_{0f} and θ_{0f} prevails. Recall that ϕ_0, ψ_0 are extended beneath $\psi_0 = 0$, with $q_0(\phi_0, \psi_0)$ and $\theta_0(\phi_0, \psi_0)$ analytically continued there (we need q_0 and θ_0 to be differentiable at $\psi_0 = 0$, as shown in the following). Thus, for the sea-water intrusion case (figure 1*b*) to be considered here, this is achieved with the aid of (3.9) extended to $\psi_0 < 0$. This is different from the problem of viscous flow past a body considered by Cole (1968), which dealt with the exterior flow $\psi_0 > 0$ only.

Similarly, the continuity equation (2.9) becomes

$$\frac{\partial}{\partial \phi_0}\left(\frac{q_\phi}{q_0}\right) + \frac{\partial}{\partial \psi_0}\left(\frac{q_\psi}{q_0}\right) = 0, \tag{4.2}$$

while the transport equation (2.10) renders

$$q_\phi \frac{\partial \rho}{\partial \phi_0} + q_\psi \frac{\partial \rho}{\partial \psi_0} = q_0\left[\frac{\partial}{\partial \phi_0}\left(\frac{F_\phi}{q_0}\right) + \frac{\partial}{\partial \psi_0}\left(\frac{F_\psi}{q_0}\right)\right]. \tag{4.3}$$

The components of the solute flux (2.11) are transformed into

$$F_\phi = q_0\left\{\left[D_0 + \varepsilon q + \varepsilon(\lambda - 1)\frac{q_\phi^2}{q}\right]\frac{\partial \rho}{\partial \phi_0} + \varepsilon(\lambda - 1)\frac{q_\phi q_\psi}{q}\frac{\partial \rho}{\partial \psi_0}\right\}, \tag{4.4a}$$

$$F_\psi = q_0\left\{\varepsilon(\lambda - 1)\frac{q_\phi q_\psi}{q}\frac{\partial \rho}{\partial \phi_0} + \left[D_0 + \varepsilon q + \varepsilon(\lambda - 1)\frac{q_\psi^2}{q}\right]\frac{\partial \rho}{\partial \psi_0}\right\} \tag{4.4b}$$

by using the relationships $\partial\rho/\partial s_\phi = q_0(\partial\rho/\partial\phi_0)$ and $\partial\rho/\partial s_\psi = q_0(\partial\rho/\partial\psi_0)$. Following Cole (1968), we switch to the normalized fluxes $w_\phi = q_\phi/q_0$, $w_\psi = q_\psi/q_0$, $w = q/q_0$ as independent variables in (4.1)–(4.4) to arrive at the final formulation

$$\frac{\partial w_\psi}{\partial \phi_0} - \frac{\partial w_\phi}{\partial \psi_0} = -\gamma \left(\frac{\cos\theta_0}{q_0} \frac{\partial \rho}{\partial \phi_0} - \frac{\sin\theta_0}{q_0} \frac{\partial \rho}{\partial \psi_0} \right), \tag{4.5}$$

$$\frac{\partial w_\phi}{\partial \phi_0} + \frac{\partial w_\psi}{\partial \psi_0} = 0, \tag{4.6}$$

$$w_\phi \frac{\partial \rho}{\partial \phi_0} + w_\psi \frac{\partial \rho}{\partial \psi_0} = \frac{\partial}{\partial \phi_0}\left(\frac{F_\phi}{q_0}\right) + \frac{\partial}{\partial \psi_0}\left(\frac{F_\psi}{q_0}\right), \tag{4.7}$$

$$F_\phi/q_0 = \left[D_0 + \varepsilon q_0 w + \varepsilon(\lambda-1) q_0 \frac{w_\phi^2}{w} \right] \frac{\partial \rho}{\partial \phi_0} + \varepsilon(\lambda-1) q_0 \frac{w_\phi w_\psi}{w} \frac{\partial \rho}{\partial \psi_0},$$

$$F_\psi/q_0 = \varepsilon(\lambda-1) q_0 \frac{w_\phi w_\psi}{w} \frac{\partial \rho}{\partial \phi_0} + \left[D_0 + \varepsilon q_0 w + \varepsilon(\lambda-1) q_0 \frac{w_\psi^2}{w} \right] \frac{\partial \rho}{\partial \psi_0}. \tag{4.8}$$

The nonlinear system (4.5)–(4.8) expresses the flow and mixing problem in terms of w_ϕ, w_ψ and ρ as functions of ϕ_0, ψ_0 with known $q_0(\phi_0, \psi_0)$ and $\theta_0(\phi_0, \psi_0)$ and given constant parameters γ, D_0, ε and λ. It must be solved under appropriate boundary conditions. Although its structure is not simpler than that of the original system (2.8)–(2.11), it permits us to formulate the BL equations in a general manner for any regular sharp interface solution.

4.1. *The boundary-layer equations*

The general approach followed by Cole (1968) for viscous flow is to expand (4.5)–(4.8) in outer and inner perturbation expansions for $D_0 = o(1)$ and $\varepsilon = o(1)$. The leading-order term of the outer expansion is the limit $D_0 = 0$ and $\varepsilon = 0$ in (4.5)–(4.8), i.e. the sharp interface solution. Assuming that the latter is defined by $\psi_0 = 0$, the solution is given in a compact form by

$$w_{\phi 0} = w_0 = H(\psi_0), \qquad w_{\psi 0} = 0, \qquad \rho_0 = 1 - H(\psi_0), \tag{4.9}$$

where H stands for the Heaviside function ($H(\psi_0) = 0$, $\psi_0 \leqslant 0$; $H(\psi_0) = 1$, $\psi_0 > 0$). The expansion is singular because of the presence of the derivatives of ρ with respect to ψ_0 in (4.7)–(4.8). Before proceeding further, we make the additional simplification of neglecting the effect of molecular diffusion term D_0 as compared to ε, since the ratio $D_0/\varepsilon = nD_0'/U\alpha_T \ll 1$. The subtle, but negligible, effect of molecular diffusion on the BL solution will be discussed later.

Following Cole (1968) we define inner, BL, variables as follows

$$\Phi = \phi_0, \qquad \Psi = \psi_0/\varepsilon^{1/2}, \qquad W_\phi = w_\phi, \qquad W_\psi = w_\psi/\varepsilon^{1/2}. \tag{4.10}$$

As an additional preparatory step we notice that near $\psi_0 = 0$ we have

$$q_0(\phi_0, \psi_0) = q_{0f}(\Phi) + \varepsilon^{1/2} \Psi \left. \frac{\partial q_0}{\partial \psi_0} \right|_{\psi_0 = 0} + \cdots \tag{4.11}$$

and similarly for $\sin\theta_0$ and $\cos\theta_0$, with $q_{0f} = q_0(\phi_0, 0)$ and $\theta_{0f} = \theta_0(\phi_0, 0)$. These are precisely the relationships used in order to continue the flow beneath $\Psi = 0$ and all that is required for the first-order BL approximation is that the normal derivatives of q_0 and θ_0 are bounded at $\psi_0 = 0$.

Substitution of (4.10) and (4.11) into (4.5), expanding in ε and retaining the leading-order term $O(1)$, the only one to be considered here, leads to

$$\frac{\partial W_\phi}{\partial \Psi} = -\frac{\gamma \sin\theta_{0f}}{q_{0f}}\frac{\partial \rho}{\partial \Psi}. \tag{4.12}$$

Since by (3.7) $q_{0f} = \gamma \sin\theta_{0f}$, equation (4.12) becomes

$$\frac{\partial W_\phi}{\partial \Psi} = -\frac{\partial \rho}{\partial \Psi}. \tag{4.13}$$

In a similar manner, (4.6)–(4.8) yield

$$\frac{\partial W_\phi}{\partial \Phi} + \frac{\partial W_\psi}{\partial \Psi} = 0, \tag{4.14}$$

$$W_\phi \frac{\partial \rho}{\partial \Phi} + W_\psi \frac{\partial \rho}{\partial \Psi} = q_{0f}\frac{\partial}{\partial \Psi}\left(W_\phi \frac{\partial \rho}{\partial \Psi}\right). \tag{4.15}$$

Assuming that the BL is developing along the interface defined by $\Psi = 0$, $\Phi > 0$ (as in figure 1), matching with the outer solution (4.9) requires

$$W_\phi \to 1,\ W_\psi \to 0,\ \rho \to 0\ \ (\text{for}\ \ \Psi \to \infty);\ \ W_\phi \to 0,\ W_\psi \to 0,\ \rho \to 1\ \ (\text{for}\ \Psi \to -\infty). \tag{4.16}$$

Similarly to Van Duijn & Peletier (1992), we further simplify the system by integrating (4.13) first, and by using (4.16) to obtain

$$W_\Phi(\Phi, \Psi) + \rho(\Phi, \Psi) = 1. \tag{4.17}$$

This result is identical to that pertaining to stratified flow (3.5) and it implies that a hydrostatic pressure distribution prevails across the boundary layer, which is consistent with the well-known property of viscous boundary layers. Equation (4.17) expresses in a simple manner the interaction between flow and density which is otherwise neglected in Darcy's law (2.1), if salt is regarded as a tracer.

Elimination of ρ leads to the final form of the BL equations for the flux vector $\boldsymbol{W}(W_\phi, W_\psi)$

$$\nabla \cdot \boldsymbol{W} = 0, \tag{4.18}$$

$$\nabla \cdot (W_\phi \boldsymbol{W}) = \frac{\partial}{\partial \Psi}\left[q_{0f}\, W_\phi \frac{\partial W_\phi}{\partial \Psi}\right], \tag{4.19}$$

where $\nabla(\partial/\partial\Phi, \partial/\partial\Psi)$ operates in the (Φ, Ψ)-plane.

Before proceeding with the approximate solution of (4.18), (4.19) we review briefly the exact results of Van Duijn & Peletier (1992).

4.2. *Review of the Van Duijn & Peletier* (*1992*) *solution of uniform flow* (*figure 1a*)

The problem is a particular case for which $q_0 = 1$, $\phi_0 = x$, $\psi_0 = y$, $W_\phi = q_x$, $W_\psi = q_y/\varepsilon^{1/2}$. There is no length scale L present in this case and Van Duijn & Peletier (1992) have solved equations similar to (4.18), (4.19), with $q_{0f} = 1$, by using the standard classical procedure of similarity variables. These are defined by $\xi = YX^{-1/2}$ and $f = \Psi X^{-1/2}$, where $X = x$, $Y = y/\varepsilon^{1/2}$, and Ψ is the streamfunction of the $\boldsymbol{W}$ field.

Although Van Duijn & Peletier (1992) have solved for the more general condition $D_0 > 0$, in the case of interest here ($D_0 = 0$) the ordinary differential equation satisfied by f was found to be

$$(f'^2)'' + ff'' = 0. \tag{4.20}$$

Equation (4.20) is different from the similar equation pertaining to laminar BL between parallel streams (Lock 1951). The difference stems from the dependence of the transverse dispersion coefficient on the flux, the leading term in (4.8) being $F_y = \varepsilon q\, \partial\rho/\partial y$, whereas in the viscous case the parallel term is linear in the velocity gradient. As a result, there is a qualitative difference between the velocity distribution in the laminar BL and the one at an interface in porous media set forth by Van Duijn & Peletier (1992). Their finding is that the lower edge of the BL is at a finite depth $\xi = -\alpha$ and there the q_x profile has a finite slope given by $\partial q_x/\partial y = c/(\varepsilon x)^{1/2}$ at $\xi \to -\alpha$ from above while $q_x \to 0$. The constant $c = 0.3551$ was determined by numerical solution of (4.20). However, if molecular diffusion is taken into account ($D_0 > 0$), the numerically determined q_x profile is continuous. For the case of practical interest for which $D_0 \ll \varepsilon$, a thin diffusive sublayer appears at $Y = -\alpha$, which is of little consequence and which we neglect here.

Another major finding of Van Duijn & Peletier (1992) is that the heavier fluid is sucked into the BL and the flux is given by

$$q_y = \varepsilon \frac{\partial q_x}{\partial y} = c\left(\frac{\varepsilon}{x}\right)^{1/2}, \quad \text{i.e.} \quad q'_y = \alpha_T \frac{\partial q'_x}{\partial y'} = c\left(\frac{\alpha_T}{x'}\right)^{1/2}. \tag{4.21}$$

Since Van Duijn & Peletier (1992) have determined the constant c for $D_0 > 0$ as well, it is of interest to examine the impact of assuming in (2.7) that the dispersion tensor D'_{ij} is constant and proportional to U. Such an assumption was adopted in previous works and it leads to linearization of the diffusive term in the right-hand side of (4.19). By a proper rescaling, it is found that the error is large, as the rate of entrainment of salt water (4.21) is almost doubled. We shall adhere here to the case of interest in applications, namely neglect of D_0 and dependence of D'_{ij} on the local flux $\boldsymbol{q}'$ as in (2.7).

The similarity variables approach does not apply to the non-uniform flow case owing to the presence of $q_{0f}(\Phi_0)$ in the right-hand side of (4.19). We proceed next with solving equations (4.18), (4.19) approximately and the result will be compared subsequently with those of Van Duijn & Peletier (1992).

5. Approximate solution of the boundary layer equations by von Kármán method

We adopt the von Kármán integral equation method as applied to the related viscous problem by Lock (1951). We refer to the sea-water intrusion problem (figure 1*b*) and we seek the solution of the BL equations (4.18), (4.19) with appropriate boundary conditions. The flow domains in the (x, y)- and (Φ, Ψ)-planes are represented in figure 2. In line with the von Kármán approach, we integrate the BL equations in the domain defined by $\Phi > 0$, $-\Delta_\ell < \Psi < \Delta_u$, where $\Psi = -\Delta_\ell(\Phi)$ and $\Psi = \Delta_u(\Phi)$ are the equations of the lower and upper edges of the BL in the (Φ, Ψ)-plane.

To formulate the boundary conditions, we observe that by (3.8) the line $\Phi = \phi_0 = 0$ represents the equipotential that passes through the origin and the velocity $q_\phi = q_0(\psi_0)$ normal to it is given by (3.9). Hence, by the definition of W_ϕ we have there $W_\phi(0, \Psi) = 1$ for $\Psi > 0$. The BL develops downstream of the origin and therefore the heavier fluid is at rest, i.e. $W_\phi(0, \Psi) = 0$ for $\Psi < 0$. At an arbitrary cross-section Φ, the unknown flux distribution $W_\phi(\Phi, \Psi)$ satisfies $W_\phi(\Phi, \Delta_u) = 1$ and $W_\phi(\Phi, -\Delta_\ell) = 0$, to match the outer solution. Along the line $\Psi = \Delta_u(\Phi)$, $W_\phi(\Phi, \Delta_u) = 1$ (consistent with the von Kármán approximation) and we define $W_\psi(\Phi, \Delta_u) = W_{\psi u}(\Phi)$, where the latter

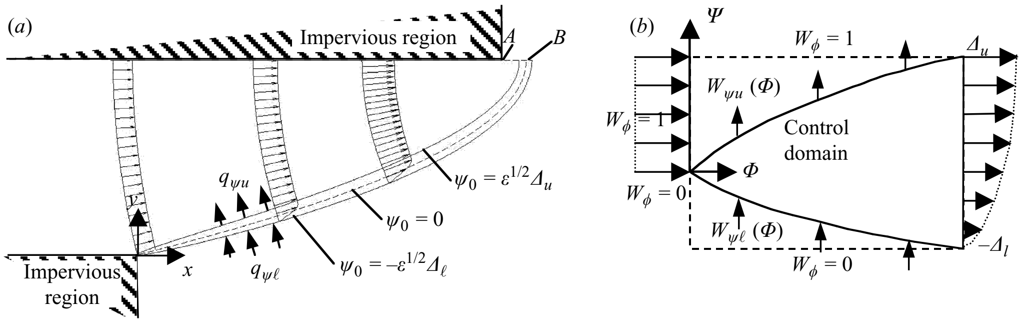

FIGURE 2. The sea-water intrusion problem. (*a*) (x, y)-plane. (*b*) (Φ, Ψ)-plane.

represents a given rate of exchange between the fresh-water body and the BL. Finally, along the BL lower edge, we have in a similar manner $W_\phi(\Phi, -\Delta_\ell) = 0$ and $W_\psi(\Phi, -\Delta_\ell) = W_{\psi\ell}(\Phi)$, the rate of salt water suction by the BL. These conditions are specified on the boundaries of the control volume in figure 2(*b*).

Integration of the BL equations (4.18), (4.19) in the control domain of figure 2(*b*) with the given boundary conditions leads to

$$-\Delta_u + \int_0^\Phi W_{\psi u}(\Phi')\,\mathrm{d}\Phi' + \int_{-\Delta_\ell}^{\Delta_u} W_\phi(\Phi, \Psi')\,\mathrm{d}\Psi' - \int_0^\Phi W_{\psi\ell}(\Phi')\,\mathrm{d}\Phi' = 0, \tag{5.1}$$

$$-\Delta_u + \int_0^\Phi W_{\psi u}(\Phi')\,\mathrm{d}\Phi' + \int_{-\Delta_\ell}^{\Delta_u} W_\phi^2(\Phi, \Psi')\,\mathrm{d}\Psi' = 0. \tag{5.2}$$

These equations are exact, the assumption that the BL upper and lower edges are at finite Ψ, notwithstanding. Differentiation with respect to Φ and elimination of $W_{\psi u}$ leads to the unique ODE

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}\int_{-\Delta_\ell}^{\Delta_u} [W_\phi(\Phi, \Psi')\,\mathrm{d}\Psi' - W_\phi^2(\Phi, \Psi')]\,\mathrm{d}\Psi' = W_{\psi\ell}(\Phi) \tag{5.3}$$

for the unknown functions W_ϕ, $W_{\psi\ell}$, Δ_ℓ and Δ_u.

An additional boundary condition is obtained by observing that the slope $\partial\rho/\partial\Psi = -\partial W_\phi/\partial\Psi$ is discontinuous at $\Psi = -\Delta_\ell$, as $\partial W_\phi/\partial\Psi > 0$ at $-\Delta_\ell^+$ and $\partial W_\phi/\partial\Psi = 0$ at $-\Delta_\ell^-$ (figure 2*b*). Hence, taking the limit of the transport equation (4.15) for $\Psi \to -\Delta_\ell^+$ we obtain

$$W_{\psi\ell} = q_{0f}\left.\frac{\partial W_\phi}{\partial\Psi}\right|_{\Psi=-\Delta_\ell^+}, \tag{5.4}$$

since $W_\phi \to 0$, while $\partial\rho/\partial\Phi$ and $\partial^2\rho/\partial\Psi^2$ are finite at $\Psi = -\Delta_\ell^+$.

This relationship is consistent with (4.21) obtained by Van Duijn & Peletier (1992) for uniform flow which expresses the balance between the vertical dispersive flux term $\alpha_T\partial/\partial y'(q'_x\,\partial\rho'/\partial y')$ and the advective term $q'_y\,\partial\rho'/\partial y'$ at the lower edge of the BL.

By the von Kármán approach, (5.3), (5.4) are further simplified by assuming a similarity solution for the velocity profile

$$W_\phi(\Phi, \Psi) = F(\zeta), \quad \zeta = \frac{\Psi + \Delta_\ell}{\Delta}, \tag{5.5}$$

where $\Delta = \Delta_\ell + \Delta_u$ is the total thickness of the BL such that $0 < \zeta < 1$ in the BL. Substitution of (5.5) into (5.3), (5.4) and differentiation leads to

$$I \frac{\mathrm{d}\Delta}{\mathrm{d}\Phi} = W_{\psi\ell}, \tag{5.6}$$

$$W_{\psi\ell} = q_{0f} \frac{F'(0)}{\Delta}, \tag{5.7}$$

where the constant I is given by

$$I = (1/\Delta) \int_{-\Delta_\ell}^{\Delta_u} [W_\phi(\Phi, \Psi') - W_\phi^2(\Phi, \Psi')] \,\mathrm{d}\Psi' = \int_0^1 [F(\zeta) - F^2(\zeta)] \,\mathrm{d}\zeta.$$

Elimination of $W_{\psi\ell}$ and integration in (5.6) yields the final solution which expresses in a simple form the dependence of Δ and $W_{\psi\ell}$ upon $q_{0f}(\phi_0)$, the flux along the sharp interface solution

$$\Delta(\Phi) = \left[\frac{2\,F'(0)}{I} K(\phi) \right]^{1/2}, \tag{5.8}$$

$$W_{\psi\ell}(\Phi) = \frac{q_{0f}\,(\Phi)\beta}{[K(\phi)]^{1/2}}, \quad \beta = \left[\frac{I\,F'(0)}{2} \right]^{1/2}, \tag{5.9}$$

where

$$K(\phi) = \int_0^\phi q_{of}(\phi') \,\mathrm{d}\phi'. \tag{5.10}$$

The von Kármán method implies selecting a suitable analytical expression for $F(\zeta)$ that satisfies the matching conditions $F(0) = 0$, $F(1) = 1$ and $F'(1) = F''(1) = \cdots = 0$ at an appropriate order, fixing therefore the values of I and $F'(0)$ in (5.8), (5.9). This solution is explored for the cases of uniform flow (figure 1*a*) and sea-water intrusion (figure 1*b*) in the next section.

While Δ and $W_{\psi\ell}$ are determined uniquely as functions of $\Phi = \phi_0$ in a simple manner for a selected shape F and for a given sharp interface solution, the relative position of the BL edges Δ_ℓ, $\Delta_u = \Delta - \Delta_\ell$ with respect to $\Psi = 0$ is left undetermined, unless $W_{\psi u}$ is specified in (5.1), (5.2). The latter represents the imposed fresh-water flux into the BL and it is equal to zero for the case of sea-water intrusion in figure 1(*b*) owing to the presence of the confining layer at $y' = b'$. This is also the condition adopted by Van Duijn & Peletier (1992) for the uniform flow in figure 1(*a*). Then, it is easily seen from (5.1) or (5.2) that

$$\frac{\Delta_u}{\Delta} = J,\ \frac{\Delta_\ell}{\Delta} = 1 - J \quad \text{with} \quad J = (1/\Delta) \int_{-\Delta_\ell}^{\Delta_u} W_\phi^2(\Phi, \Psi') \,\mathrm{d}\Psi' = \int_0^1 F^2(\zeta) \,\mathrm{d}\zeta. \tag{5.11}$$

These findings are consistent with those of Lock (1951) for viscous shear layers. Finally, to determine the BL location in the physical plane x, y, we must project $\psi_{0u} = \varepsilon^{1/2} \Delta_u(\phi_0)$ and $\psi_{0\ell} = -\varepsilon^{1/2} \Delta_\ell(\phi_0)$ by using the mapping (3.8) or similar ones, depending on the sharp interface solution. Similarly, the flux of salt water into the BL at the lower edge is given by $q_\psi = \varepsilon^{1/2} q_0(\phi_0, \psi_{0\ell}) W_{\psi\ell}(\phi_0)$, which at the leading order can also be written as $q_\psi = \varepsilon^{1/2} q_{0f}(\phi_0) W_{\psi\ell}(\phi_0)$, with q_ψ normal to the sharp interface. By using this last expression, we can determine the total salt-water discharge between the toe and any point along the interface by

$$Q_s(\phi_0) = \int_0^s q_\psi\, ds_\phi = \varepsilon^{1/2} \int_0^{\phi_0} W_{\psi\ell}(\Phi'') \,\mathrm{d}\Phi'', \tag{5.12}$$

where we used the relationship $\mathrm{d}s_\phi = \mathrm{d}\phi_0/q_{0f}$. By using (5.9), we obtain

$$
\begin{aligned}
Q_s(\phi_0) &= \varepsilon^{1/2}\beta \int_0^{\phi_0} q_{0f}(\phi_0')[K(\phi_0')]^{-1/2}\,\mathrm{d}\phi_0' \\
&= \varepsilon^{1/2}\beta \int_0^{\phi_0} \frac{\mathrm{d}K}{\mathrm{d}\phi_0'} K^{-1/2}\,\mathrm{d}\phi_0' \\
&= 2\varepsilon^{1/2}\beta\,[K(\phi_0)]^{1/2}.
\end{aligned} \tag{5.13}
$$

The flux profile within the BL is obtained in a similar manner from $q_\phi = q_0(\psi_0)\,F(\zeta)$, with ζ given by (5.5) and $\psi_0 = \varepsilon^{1/2}\Psi$ varying between $-\varepsilon^{1/2}\Delta_\ell$ and $\varepsilon^{1/2}\Delta_u$.

5.1. *First-order correction to the outer flow*

In the case of viscous flow past a body, Cole (1968) determined in a systematic manner the next correction to the outer flow w_1. The well-known result is that the correction manifests itself as a potential flow with boundary condition $w_{\psi 1}(\phi_0, 0) = \varepsilon^{1/2} W_\Psi(\Phi_0)$, which can be interpreted as flow due to the displacement thickness.

In the present case, since there is no through-flow at the upper edge of the BL, the correction applies to the salt-water body only. The correction in the physical plane x, y is given by $\boldsymbol{q}_1(x, y)$ which satisfies

$$
\boldsymbol{q}_1 = \nabla\phi_1, \quad \nabla^2\phi_1 = 0 \quad \text{for} \quad \psi_0 < 0, \tag{5.14}
$$

with boundary condition

$$
q_{\psi 1} = \varepsilon^{1/2} q_{of}(\phi_0)\, W_{\Psi\ell}(\phi_0) \quad \text{at} \quad \psi_0 = 0, \tag{5.15}
$$

with $W_{\Psi\ell}(\phi_0)$ given by (5.9). In other words, the potential flow in the salt-water body is caused by the given flux (5.15), normal to the sharp interface. Unlike the BL problem, in order to solve for ϕ_1 we need to know the shape of the domain of the salt-water body and the appropriate conditions on the boundaries, besides (5.15).

6. Application to uniform flow and to sea-water intrusion problems and comparison with existing solutions

6.1. *Uniform flow* (*figure 1a*)

This is a particular case of the general approximate BL solution of the preceding section for $q_0 = q_{0f} = 1$, $\phi_0 = x$, $\psi_0 = y$. It is easy to compare the approximate results based on the von Kármán method with the exact result of Van Duijn & Peletier (1992) for the salt-water flux (4.21) $q_y = 0.355(\varepsilon/x)^{1/2}$.

Thus, by selecting for the flux profile the simple parabolic shape $F(\zeta) = 2\zeta - \zeta^2$ that satisfies $F(0) = F'(1) = 0$, $F(1) = 1$, we obtain $I = 2/15$, $J = 8/15$, $F'(0) = 2$, $\beta = 0.365$. Substitution in (5.9) with $q_{0f} = 1$, $\Phi = x$ leads to $q_y = W_{\psi\ell} = 0.365(\varepsilon/x)^{1/2}$, which exceeds the exact solution by 4 %.

Adopting the fourth-order polynomial proposed by Lock (1951) for viscous shear flow, namely $F(\zeta) = 2\zeta - 2\zeta^3 + \zeta^4$ ($F'(1) = F''(1) = 0$), the result is

$$
I = 37/315, \qquad J = 367/630, \qquad F'(0) = 2, \qquad \beta = 0.343, \tag{6.1}
$$

and (5.9) leads to $q_y = W_{\psi\ell} = 0.343(\varepsilon/x)^{1/2}$ which is smaller than the exact result by 2 %. We shall adopt this profile for further computations.

We conclude that the approximate BL solution based on the von Kármán method leads to accurate results in the case of uniform flows. We explore it next for solving the problem of non-uniform flow, the main topic of the present study.

6.2. *Sea-water intrusion* (*figure 1b*)

With the Glover (1959) approximation adopted for the zero-order outer flow (the sharp interface solution) it is possible to derive the boundary solution (5.8)–(5.11) in an analytical closed form. Indeed, with flux along the interface $q_{0f}(\phi_0)$ given by (3.9) we obtain

$$K(\phi) = \int_0^{\phi} \frac{\mathrm{d}\phi'}{[(1-\eta\phi')^2+\eta^2]^{1/2}} = \frac{1}{\eta}\left[\operatorname{arcsinh}\left(\phi-\frac{1}{\eta}\right)+\operatorname{arcsinh}\left(\frac{1}{\eta}\right)\right]. \qquad (6.2)$$

Recall that the parameter $\eta = U/[(\kappa g/\mu)(\rho_s/\rho_f - 1)] < 0.5$ characterizes the 'flatness' of the sharp interface, since the dimensionless distance to the toe is $x_B = 1/(2\eta) > 1$ (figure 1*b*).

Hence, we obtain from (5.8), (5.9) the BL thickness and the dimensionless salt-water flux, respectively, as functions of $\Phi = \phi_0$ along the interface $\psi_0 = 0$. Thus, it is possible to depict the upper and lower edges of the BL (5.11) $\psi_0 = \varepsilon^{1/2}\Delta_u(\phi_0)$ and $\psi_0 = -\varepsilon^{1/2}\Delta_\ell(\phi_0)$, respectively, in the physical plane x, y by using the mapping (3.8) for $0 < \phi_0 < 1/\eta$. In particular, the maximal thickness is reached at the outflow face AB (figure 1*b*) for $\phi_0 = 1/\eta$, i.e. $\Delta_{max} = \Delta(1/\eta) = [2F'(0)\,K(1/\eta)/I]^{1/2} = [2F'(0)\operatorname{arcsinh}(1/\eta)/\eta I]^{1/2}$. In order to make the BL approximation a valid one, $\psi_0 = \varepsilon^{1/2}\Delta_{umax} = \varepsilon^{1/2} J \Delta_{max}$ must be much smaller than unity, such as not to alter significantly the flow at AB (figure 1*b*).

The analytical expression of $W_{\psi\ell}(\phi_0)$ (5.9) can be used in (5.15) in order to derive the first-order correction $\boldsymbol{q}_1$ to flow in the salt-water body.

One of the variables of interest is the discharge of salt water entrained in the BL (5.12). It is determined by substituting (6.2) into (5.13), and the total discharge to the sea is given by $Q_{s\,max} = Q_s(1/\eta)$. Recall that the dimensionless discharge is defined by $Q_s = Q_s'/Ub' = Q_s'/Q'$, the ratio between the discharge of salt water and that of fresh water. Substitution of K (6.2) into (5.13) yields the final expressions

$$Q_s = 2\varepsilon^{1/2}\beta\left\{\frac{1}{\eta}\left[\operatorname{arcsinh}\left(\phi_0-\frac{1}{\eta}\right)+\operatorname{arcsinh}\left(\frac{1}{\eta}\right)\right]\right\}^{1/2}, \qquad (6.3a)$$

$$Q_{s\,max} = 2\varepsilon^{1/2}\beta\left[\frac{1}{\eta}\operatorname{arcsinh}\left(\frac{1}{\eta}\right)\right]^{1/2}. \qquad (6.3b)$$

In order to illustrate the results, we have plotted figure 3(*a*) the dependence of $Q_s/(\varepsilon^{1/2}\beta)$ (6.3*a*) upon the ratio s/x_B, where s is the dimensionless distance s from the toe $\phi_0 = 0$ along the sharp interface $\psi_0 = 0$. The relationship $\mathrm{d}s = (\mathrm{d}x^2 + \mathrm{d}y^2)^{1/2}$ and (3.8) were used for this purpose, while $x_B = 1/(2\eta)$. The curves are depicted for different values of η and they can be readily used for estimating the marine-water entrainment along the BL. Since the BL approximation is supposed to be valid if $Q_{s\,max} \ll 1$, we may estimate from (6.3*b*) what are the maximal values of admissible ε for a given η. Thus, for $\eta = 0.1$ (i.e. $x_B = 5$), $\beta = 0.343$ and $Q_{s\,max} = 0.05$, we arrive at $\varepsilon_{max} \simeq 1.8 \times 10^{-4}$ while for $\eta = 0.5$ ($x_B = 1$), $\varepsilon_{max} \simeq 2 \times 10^{-3}$. These values are within the expected range in field applications.

In order to further illustrate the results, we compare them with published numerical solutions. The problem of sea-water intrusion was solved numerically first by Henry (1964), and the solution has served as a benchmark for many other numerical works. However, Henry (1964) has assumed that the dispersion coefficients are constant and isotropic, i.e. in our notation he has solved the problem with $D'_{ij} = \alpha U\,\delta_{ij}$ in (2.7). Besides, the Péclet number was quite small, $Pe = 10$, and the boundary condition at

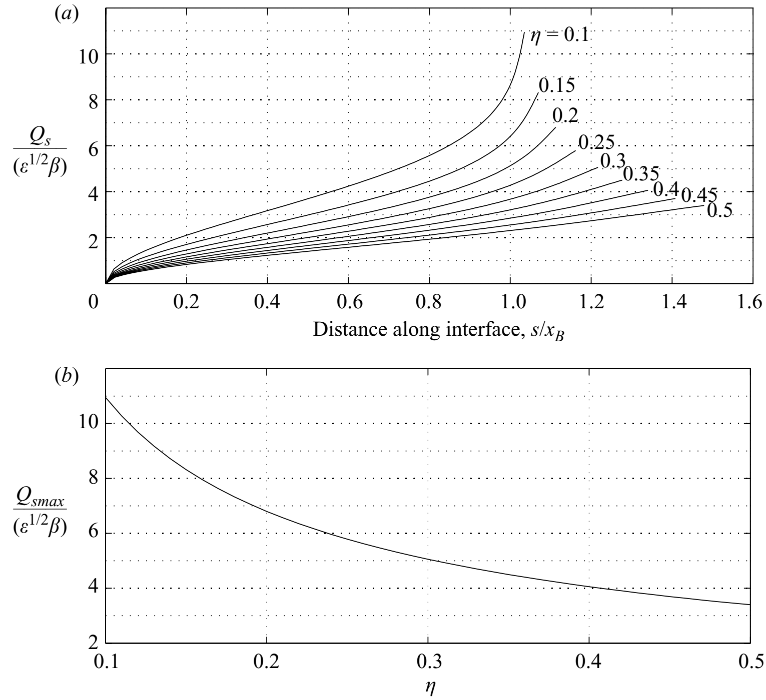

FIGURE 3. (*a*) Ratio of the accumulated salt-water entrainment to the total fresh-water discharge, for different values of η, as a function of the distance along the interface. The ratio at the sea outflow face as a function of η is given in (*b*).

the coast was different. Thus, the solution is not directly comparable with the present one.

Smith (2004) solved numerically the problem of sea-water intrusion, including the effect of density and the dependence of the dispersion coefficients on the flux, as in (2.7). Comparison with his simulations is limited by his choice of values for $\varepsilon = \alpha_T/b' = 1/Pe$, the minimal one being 1/200, much larger than that anticipated in applications, and preventing the comparison with the BL approximation used here. Nevertheless, we have adopted this value for the purpose of comparison, for the case with the largest considered value of $\eta = 0.444$. The numerically determined value of $Q_{s\,max}$ was 0.079 whereas the BL solution (6.3*b*) led to $Q_{s\,max} = 0.091$, a difference of 14 %. This may be regarded as good agreement in view of the large ε and the difference in flow patterns in the toe region. Smith (2004) has solved for a horizontal bottom and, in such a case, the sharp interface and BL solutions cannot represent accurately the flow pattern beneath the interface near the origin. This local effect may also explain the seaward deviation of the toe (defined by the intersection of the line $\rho = 0.5$ with the aquifer bottom) from that predicted by Glover (1959).

To further illustrate the approach we show in figure 4(*a*) the BL development along the interface and a few flux profiles for the parameter values $\varepsilon = 10^{-4}$ and $\eta = 1/4$ ($x_B = 2$). This was achieved by using (5.8) to compute Δ, and the fourth-order polynomial for $F(\zeta)$. Subsequently, the first-order correction of the salt-water flow from the sea toward the interface is shown in figure 4(*b*), by assuming that the impervious bottom is horizontal and the aquifer extends beneath the sea to $x = 5x_B = 10$. The potential $\phi_1(x, y)$ (5.14) and the associated streamfunction were determined by

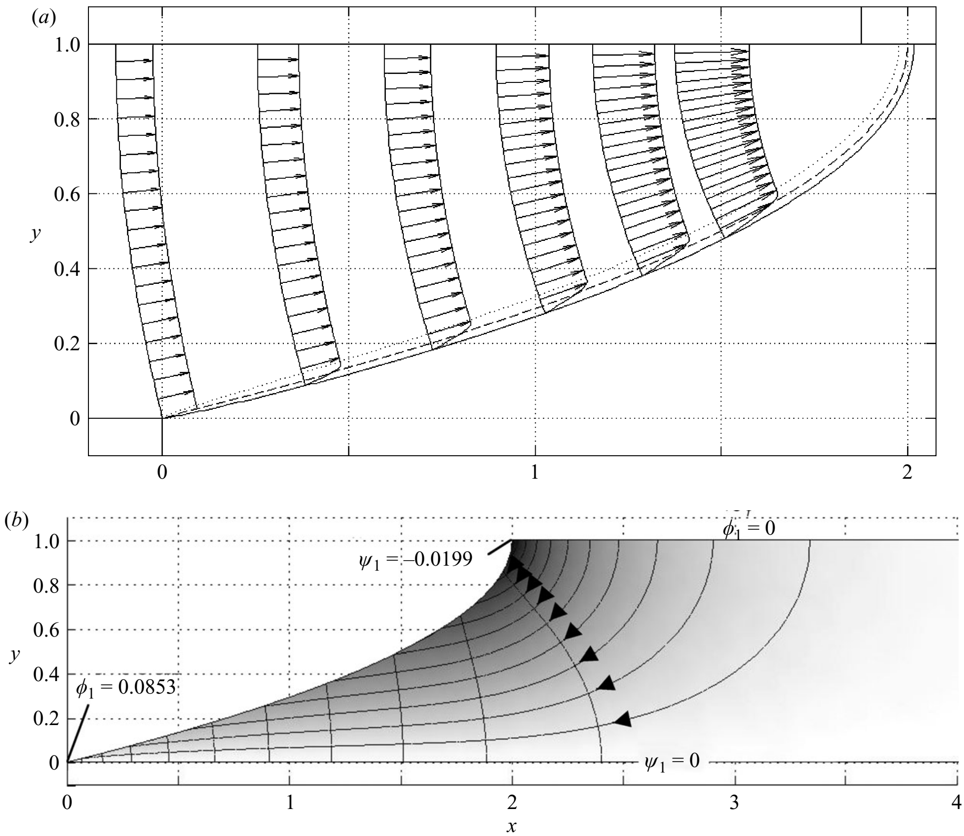

FIGURE 4. (a) Boundary-layer development and flux profiles in the fresh-water zone and in the boundary layer ..., upper edge of BL; - - -, sharp interface; ——, lower edge of BL. (b) Potential and streamfunction of the salt-water flow from the sea towards the interface (first-order correction). Both figures are for $\varepsilon = 10^{-4}$ and $\eta = 1/4$, with fourth-order polynomial assumed for $F(\zeta)$.

numerically solving the Laplace equation by a standard code, with the boundary condition of the given flux (5.15), (5.9) along the interface, of constant potential on the sea bottom and no flow at $x = 10$. The pattern is in qualitative agreement with that determined by Smith (2004) by a numerical solution of the equations of flow.

7. Summary and conclusions

We consider steady and stable flow of a lighter fluid (fresh water) above a body of heavier fluid (salt water) in porous media. In the absence of mixing, a sharp interface separates the two fluids, the lower one being at rest. Both flux and density are discontinuous across the interface. In reality, pore-scale dispersion causes mixing and, as a result, a transition zone develops along the interface. The pertinent equations of flow are nonlinear owing to the variable density and the dependence of the dispersion coefficients on flux. This is different from the similar problem of shear laminar flow between two streams in which mixing is governed by the viscosity coefficient, which does not depend on velocity.

The relevant Péclet number of the problem $Pe = L/\alpha_T$, where L is a characteristic length (i.e. formation thickness) and α_T is the transverse dispersivity, is much larger than unity in most applications. As a consequence, the thickness of the mixing layer is small compared to L. Such cases cannot be solved by most numerical codes and approximate solutions available in the hydrological literature, which were devised for moderate or small Pe. To tackle the high Pe flows, we have adopted the BL (boundary-layer) approximation, that was used by Lock (1951) for solving the viscous shear-flow problem and by Van Duijn & Peletier (1992) for flow in porous media. The latter have solved the problem of fresh water moving at uniform flux distribution, above a planar interface. The main contribution of the present study is to extend the approach to non-uniform flow of the lighter fluid, as encountered in applications. Towards this aim, we have followed the Cole (1968) approach to viscous BL, in which the velocity is regarded as a function of the potential and streamfunction of the fresh-water interface flow, as independent variables. Subsequently to formulating the BL equations, we have solved them approximately by adopting the von Kármán integral method (Lock 1951).

One of the important findings of Van Duijn & Peletier (1992) was that the heavier fluid is entrained by the mixing layer, causing flow in the lower fluid. For uniform flow, we have compared the results based on the von Kármán approximation with those obtained by Van Duijn & Peletier (1992) who numerically integrated the complete BL equations, and the agreement between the entrainment rates was excellent.

We have applied the approach to the classical and important problem of salt-water intrusion in coastal aquifers. We have been able to derive the BL thickness, the flux distribution, the rate of entrainment and the flow pattern in the salt-water body in a general manner. In contrast, the relative position of the BL with respect to the sharp interface depends on the rate of extraction or of recharge of fresh water. The latter was assumed to be zero for the problem at hand.

Concluding, it seems that realistic modelling of problems of variable density flows in porous media encountered in applications, calls for development of numerical tools that can account for the existence of a thin transition zone. The BL approach may be an appropriate one and the present results may serve as a benchmark for more complex problems.

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